The k-Fold Matroid Secretary Problem

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Setup

Elements $N \coloneqq [n]$

Weight function $w : N \to \mathbb{R}_+$, also define $w(S) \coloneqq \sum_{i \in S} w(i)$

Matroid $\mathcal{M} \subseteq 2^N$

Matroids

Set family satisfying

- Downward-closure: if $S \in \mathcal{M}$, so are all subsets of S
- Exchange property: for all $S \in \mathcal{M}$ and i, there exists j s.t. $S + i - j \in \mathcal{M}$

Example: **k–uniform matroid** consists of all sets of size at most k

i **improves** $S \in \mathcal{M}$ w.r.t. $\mathcal{M}$ if there exists j s.t. $S + i - j \in \mathcal{M}$ and $w(S + i - j) \geq w(S)$

k-Fold Matroid Unions

Given a matroid $\mathcal{M}$, its **k-fold union** $\mathcal{M}^k$ consists of all sets formed by the union of any k sets in $\mathcal{M}$

- E.g., the k-fold union of the 1-uniform matroid is the k-uniform matroid

Matroid Secretary Problem

Unknown weights $w : N \to \mathbb{R}_+$ and known matroid $\mathcal{M} \subseteq 2^N$

Elements arrive in uniformly random order

When element i arrives, learn $w(i)$

Decide to accept or reject i, subject to accepted elements in $\mathcal{M}$

Objective: maximize weight of accepted set

Competitive Ratio

An algorithm is ***α*-competitive** if in expectation, weight of accepted set is within an α factor of best possible weight

Unknown if $\Omega(1)$ competitive ratio possible (huge open problem)

For ***k*-uniform** matroids, $1-\Theta\left(\sqrt{1/k}\right)$ competitive ratio is optimal

Theorem: $1 - O\left(\sqrt{\log(n)/k}\right)$ competitive ratio for k-fold unions

Algorithm

Intuition: k-Uniform Matroids

In k-uniform matroids, want to select the k highest weight elements

Thresholding rule: in second half, accept every element which is better than the $k/2$-best element from first half

The $k/2$-best element from first half is the $\left(1 \pm o(1)\right)k$-best overall

Generalizing

The "space occupied" by S in a k-uniform matroid is just $|S|$, an additive function; concentrates and has well-behaved expectation

The analogue in a k-fold matroid $\mathcal{M}^k$ is the **covering number** $\varphi(S)$, the minimum number of sets from $\mathcal{M}$ needed to cover S

How to analyze φ?

Nash-Williams Theorem

$$\varphi(S) = \max_{T \subseteq S} \frac{|T|}{\operatorname{rank}(T)} = \max_{F \subseteq \mathcal{F}} \frac{|S \cap F|}{\operatorname{rank}(F)}$$

Can rewrite as a max of additive functions

Upper tail can be controlled with Chernoff + union bound

Algorithm for k-Fold MSP

Call the first half of elements S and the second half of elements T

Let S^* be the max-weight set of $\mathcal{M}^{k/2}$, using only elements of S

Let T^* be the elements from T which improve S^* w.r.t. $\mathcal{M}^{k/2}$

Algorithm for k-Fold MSP

Call the first half of elements S and the second half of elements T

Let S^* be the max-weight set of $\mathcal{M}^{k/2}$, using only elements of S

Let T^* be the elements from T which improve S^* w.r.t. $\mathcal{M}^{k/2}$

Accept everything from T^* so long as the accepted set $\mathrm{ALG} \in \mathcal{M}^{k/2}$

(Recursively run algorithm on S)

Analysis

Analysis

S^* is max-weight set of $\mathcal{M}^{k/2}$, using only elements of S

T^* is elements from T which improve S^* w.r.t. $\mathcal{M}^{k/2}$

ALG is elements from T accepted by the algorithm

Analysis

S^* is max-weight set of $\mathcal{M}^{k/2}$, using only elements of S

Let S^+ be S intersected with the overall max-weight set of $\mathcal{M}^k$

T^* is elements from T which improve S^* w.r.t. $\mathcal{M}^{k/2}$

ALG is elements from T accepted by the algorithm

Analysis

If $\varphi(S^{+}) \leq k/2$, then $w(S^{+}) \leq w(S^{*})$

If $\varphi(T^{*}) \leq k/2$, then $\mathrm{ALG} = T^{*}$

Our upper tail bound says that the above hold w.h.p.

Therefore, $\mathbb{E}[w(S^{+})] \lesssim \mathbb{E}[w(S^{*})]$ and $\mathbb{E}[w(T^{*})] \approx \mathbb{E}[w(\mathrm{ALG})]$

Analysis

S^* equivalent to “elements from S which improve S^*”

T^* is defined as “elements from T which improve S^*”

Elements equally likely to be in S or T, so $\mathbb{E}[w(S^*)] = \mathbb{E}[w(T^*)]$

Therefore, $\mathbb{E}[w(S^+)] \lesssim \mathbb{E}[w(S^*)] = \mathbb{E}[w(T^*)] \approx \mathbb{E}[w(\mathrm{ALG})]$

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Therefore, $\mathbb{E}[w(S^+)] \lesssim \mathbb{E}[w(S^*)] = \mathbb{E}[w(T^*)] \approx \mathbb{E}[w(\mathrm{ALG})]$

So, $\mathrm{OPT}/2 = \mathbb{E}[w(S^+)] \lesssim \mathbb{E}[w(\mathrm{ALG})] =$ second half performance

Open Questions

We achieve $1 - O\left(\sqrt{\log(n)/k}\right)$ competitive ratio

For k-uniform matroids, $1 - \Theta\left(\sqrt{1/k}\right)$ is right answer

We get utility-competitive; can we get probability-competitive?

Thank You!

QUESTIONS?